Towards elastoplastic topology optimization with direct methods

Mathilde Boissier, Georgios Michailidis, Natasha Vermaak

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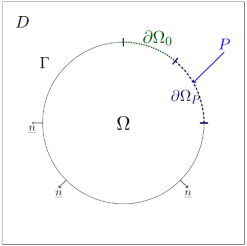
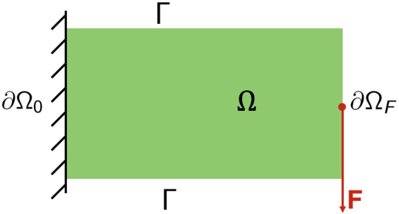
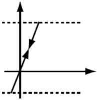
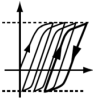
Introduction: context of this project



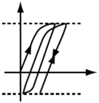




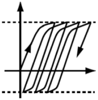
Elasticity:



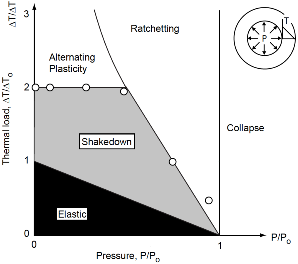
Alternative Plasticity: low-cycle fatigue



Shakedown:



Rattcheting: low-cycle fatigue



Introduction: context of this project

Project: minimize the weight of a structure under shakedown: make the solid as light as possible still keeping a shakedown behaviour.

Problem settings

Solid Ω elastoplastic

- boundary $\partial\Omega = \partial\Omega_F \cup \partial\Omega_0 \cup \Gamma$ of normal $\underline{n}$
  - $\partial\Omega_F$: non-optimizable
  - $\partial\Omega_0$: partially optimizable
  - Γ: optimizable

$$\partial\Omega_0 \cap \partial\Omega_F = \emptyset;\ \partial\Omega_F \cap \Gamma = \emptyset;\ \Gamma \cap \partial\Omega_0 = \emptyset$$

- elastoplasticity characterised by the Von Mises function f and the yield stress σ_Y by the elastoplastic stresses: $\{\sigma \text{ s.t. } f(\sigma) \leq \sigma_Y\}$

Problem reductions

- 2 dimensions.
- Loading: one cyclic load F, cycling between 0 and F_{max}.

Framework

- **The shakedown constraint**
- Optimization algorithm
- Preliminary results and future work

The shakedown constraint

Characterizing shakedown [21]:

- Following the loading history
- **Direct methods**

$\Longrightarrow$ Lower-Bound Theorem (Melan, Koiter, Köenig)

Condition for shakedown [13,12,11]:

A structure, Ω, will shakedown if there exists a stress field, σ, such that:

$$\begin{array}{ll} \int_\Omega (\sigma - \sigma^e(\Omega)) : e(\zeta) \mathrm{d}\Omega = 0 & \forall \zeta \in V \\ \forall \underline{x} \in \Omega \qquad f\left(\sigma - \sigma^e(\Omega)\right)(\underline{x}) \leq \sigma_Y & \\ \qquad\qquad f\left(\sigma\right)(\underline{x}) \leq \sigma_Y & \end{array} \tag{1}$$

with $\sigma^e(\Omega)$ the fictitious elastic stress caused by the load F_{max} and

$$V = \left\{ v \in H^1(\Omega)^d \; st \; v = 0 \; on \; \partial\Omega_0 \right\} \tag{2}$$

The shakedown constraint

Difficulties: σ doesn't result from an easy PDE formulation (not even unique).

Idea: Set σ as an unknown:

$$\min_{\substack{\Omega \in U \\ \sigma,\, \sigma^e \in C^0(D, \mathcal{S}_2(\mathbb{R}))}} \quad J(\Omega, \sigma) = \int_\Omega \mathrm{d}\Omega$$

$$st \begin{cases} \int_\Omega (\sigma - \sigma^e) : e(\zeta) \mathrm{d}\Omega = 0 \qquad \forall \zeta \in V \\ \forall \underline{x} \in \Omega \qquad \begin{array}{l} f\left(\left(\sigma - \sigma^e\right)\left(\underline{x}\right)\right) \leq \sigma_Y \\ f\left(\sigma\right)\left(\underline{x}\right) \leq \sigma_Y \end{array} \\ \sigma^e \text{ the fictitious elastic stress caused by the load } F_{max} \end{cases} \tag{3}$$

The shakedown constraint

Simplifications

- Pointwise condition ($\forall x$) hard to consider $\Rightarrow$ global constraints:

$$\begin{aligned} &\int_\Omega \left(f\left(\sigma - \sigma^e\right)\left(\underline{x}\right) - \sigma_Y\right) \mathrm{d}\Omega \leq 0 \\ &\int_\Omega \left(f\left(\sigma\right)\left(\underline{x}\right) - \sigma_Y\right) \mathrm{d}\Omega \leq 0 \end{aligned} \tag{4}$$

- Avoiding a shape with no solid [14]: minimize until volume=0
change the objective function adding the compliance*small coefficient
$\Rightarrow$ rigidity still needed:

$$J(\Omega) = \int_\Omega \mathrm{d}\Omega + l * \int_\Omega Ae(u^e : e(u^e))\mathrm{d}\Omega \tag{5}$$

The shakedown constraint

Simplifications

- fictitious elastic displacement u^e results from a PDE equivalent to a variational problem:

$$\forall v \in V = \{v \in H^1(\Omega)\, st\, v = 0\, on\, \partial\Omega_0\}$$
$$\int_\Omega Ae(u^e(\Omega)) : e(v)\mathrm{d}\Omega = \int_{\partial\Omega_F} F_{max} v \mathrm{d}s \tag{6}$$

Stress found using Hooke's law: $\boldsymbol{\sigma^e = Ae(u^e)}$.
With:

$$A\xi = 2\mu\xi + \lambda(\mathrm{Tr}\xi)I_2$$
$$e : u \to e(u) = \begin{matrix} \partial_x u_x & \frac{(\partial_x u_y + \partial_y u_x)}{2} \\ \frac{(\partial_x u_y + \partial_y u_x)}{2} & \partial_y u_y \end{matrix}$$

Introducing the solution $u^e(\Omega)$ and $\sigma^e(\Omega)$ of this elastic problem $\Rightarrow$ elimination of the elastic constraint.

The shakedown constraint

Final formulation

$$min\, J(\Omega,\sigma) = \int_\Omega \mathrm{d}\Omega + l * \int_\Omega Ae(u^e(\Omega) : e(u^e(\Omega))\mathrm{d}\Omega$$
$$\Omega \in U$$
$$\sigma \in C^0(D, \mathcal{S}_2(\mathbb{R}))$$

Objective Function: volume+l*compliance

$$st \quad \begin{cases} \int_\Omega (\sigma - \sigma^e(\Omega)) : e(\zeta)\mathrm{d}\Omega = 0 \qquad \forall \zeta \in V \\ \int_\Omega (f(\sigma) - \sigma_Y)\,\mathrm{d}\Omega \leq 0 \\ \int_\Omega (f(\sigma - \sigma^e(\Omega)) - \sigma_Y)\,\mathrm{d}\Omega \leq 0 \end{cases}$$

self-equilibrating condition

averaged safe-state conditions

with:

$$\int_\Omega Ae(u^e(\Omega)) : e(v)\mathrm{d}\Omega = \int_{\partial\Omega_F} F_{max} v \mathrm{d}s \quad \forall\, v \in V$$

elastic variational problem

Framework

- The shakedown constraint
- **Optimization algorithm**
- Preliminary results and future work

Optimization algorithm

$$\min_{\substack{\Omega \in U \\ \sigma \in C^0(D, \mathcal{S}_2(\mathbb{R}))}} \quad J(\Omega, \sigma) = \int_\Omega \mathrm{d}\Omega + l * \int_\Omega Ae(u^e(\Omega) : e(u^e(\Omega))\mathrm{d}\Omega$$

$$st \quad \begin{cases} \int_\Omega (\sigma - \sigma^e(\Omega)) : e(\zeta)\mathrm{d}\Omega = 0 \qquad \forall \zeta \in V \\ \\ \int_\Omega \left(f(\sigma) - \sigma_Y\right) \mathrm{d}\Omega \leq 0 \\ \\ \int_\Omega \left(f(\sigma - \sigma^e(\Omega)) - \sigma_Y\right) \mathrm{d}\Omega \leq 0 \end{cases}$$

$$\text{with} \quad \int_\Omega Ae(u^e(\Omega)) : e(v)\mathrm{d}\Omega = \int_{\partial\Omega_F} F_{max} v \mathrm{d}s \quad \forall\, v \in V$$

2 issues:

- Dealing with the constraints: Augmented Lagrangian Method
- Updating the shape: Level-set method

Optimization algorithm

Augmented Lagrangian Method (ALM)[5,15]

- Type of problems considered

$$\min_x f(x) \qquad st \begin{cases} c_e(x) = 0 \\ c_i(x) \geq 0 \end{cases} \tag{7}$$

- Penalization of contraints: transformed problem

$$\min_x L^{ALM}(x, \lambda_e, \lambda_i; \mu) = f(x) - \sum_l \lambda_e^l c_e^l(x) + \frac{1}{2\mu} \sum_l (c_e^l(x))^2 + \sum_j \psi(c_i^j(x), \lambda_i^j; \mu) \tag{8}$$

$$with\ \psi(t, \sigma; \mu) = \begin{cases} -\sigma t + \frac{1}{2\mu} t^2 & if\ t - \mu\sigma \leq 0 \\ -\frac{\mu}{2}\sigma^2 & otherwise \end{cases}$$

- Updating the multipliers (with x_k minimizer of L^{ALM} at iteration k.)

$$\lambda_e^{k+1} = \lambda_e^k - c_e(x_k)\mu; \qquad \lambda_i^{k+1} = max\left(\lambda_i^k - \frac{c_i(x_k)}{\mu}\right) \tag{9}$$

Optimization algorithm

Dealing with the constraints

- inequality constraints handled with to the ALM.
- equality constraint not handled with the ALM: ζ is already a Lagrange multiplier, giving the required degree of freedom.

New optimization problem

- New objective function

$$
\begin{aligned}
L^{ALM}(\Omega, \sigma, \lambda_1, \lambda_2, \zeta; \mu) = & \int_\Omega \mathrm{d}\Omega + l * \int_\Omega Ae(u^e(\Omega) : e(u^e(\Omega))\mathrm{d}\Omega \\
& + \int_\Omega (\sigma - \sigma^e(\Omega)) : e(\zeta)\mathrm{d}\Omega + \psi\left(-\int_\Omega (f(\sigma) - \sigma_Y)\,\mathrm{d}\Omega, \lambda_1; \mu\right) \\
& + \psi\left(-\int_\Omega (f(\sigma - \sigma^e(\Omega)) - \sigma_Y)\,\mathrm{d}\Omega, \lambda_2; \mu\right)
\end{aligned}
\tag{10}
$$

- New optimization problem

$$
\min_{\Omega, \sigma, \lambda 1, \lambda 2, \zeta} \quad L^{ALM}(\Omega, \sigma, \lambda_1, \lambda_2, \zeta; \mu) \tag{11}
$$

Optimization algorithm

Karush-Kuhn-Tucker (KKT) necessary condition for an optimum:

If $X = (\Omega, \sigma, \lambda_1, \lambda_2, \zeta)$, KKT condition: $\nabla L^{ALM}(X) = 0$

$$\begin{matrix} \partial_\Omega L^{ALM} = 0 & \partial_{\lambda_1} L^{ALM} = 0 & \partial_{\lambda_2} L^{ALM} = 0 \\ \partial_\sigma L^{ALM} = 0 & \partial_\zeta L^{ALM} = 0 & \end{matrix} \tag{12}$$

Iterative algorithm to make L^{ALM} decrease (at iteration k)

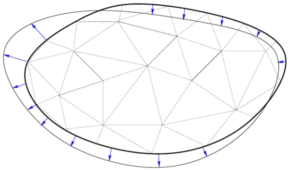
$$\begin{aligned}
&\sigma^{k+1} = \sigma^k - \partial_\sigma L^{ALM}(X^k) * step, \\
&\zeta^{k+1} = \zeta^k - \partial_\zeta L^{ALM}(X^k) * step, \\
&\lambda_1^{k+1} = max\left(\lambda_1^k + \frac{1}{r}\int_\Omega (f(\sigma) - \sigma_0)\,\mathrm{d}\Omega, 0\right) \quad \text{(from the ALM)}, \\
&\lambda_2^{k+1} = max\left(\lambda_2^k + \frac{1}{r}\int_\Omega (f(\sigma - \sigma^e(\Omega)) - \sigma_0)\,\mathrm{d}\Omega, 0\right) \quad \text{(from the ALM)}, \\
&\Omega^{k+1} \,\text{actualized by the level-set method with velocity V} \\
&\qquad with \quad \partial_\Omega L^{ALM}(X^k) = \int_{\partial\Omega} V\theta.nds
\end{aligned} \tag{13}$$

Optimization algorithm

Level-set method (2,3,4,14)

Shape, Ω, represented by a function ϕ:

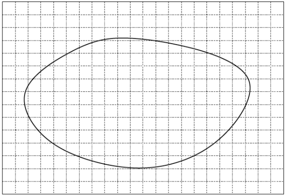
$$\begin{cases} \phi(x) = 0 & x \in \partial\Omega \cap D \\ \phi(x) < 0 & x \in \Omega \\ \phi(x) > 0 & x \in D \setminus \Omega \end{cases} \qquad (14)$$



Shape tracking

Evolution of ϕ: Hamilton-Jacobi equation

$$\partial_t \phi + V \mid \nabla\phi \mid = 0 \qquad (15)$$



Shape capturing

No need to remesh. At each iteration, "picture" of the shape

$\Rightarrow$ Shape Capturing

Velocity, V, determined by the objective function's shape derivative.

Algorithm

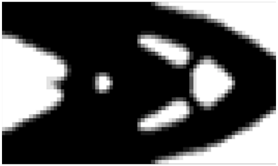
- Set the Finite Element Formulation
- Choose all the parameters
  - Model: elasticity parameters, ...
  - Finite Element Formulation: number of elements, ...
  - Optimization coefficient: maximum number of iterations, ...
  - Coefficient for the compliance l
- Initialize the variables: X^0
- Iterative algorithm:
  - while $\|L_{old}^{ALM} - L^{ALM}\| > precision$ do
    - choose a step
    - from iteration k, compute the variables at k+1
  - end do
- check that the final shape satisfies the shakedown conditions.

Framework

- The shakedown constraint
- Optimization algorithm
- **Preliminary results and future work**

Preliminary results and future work

Elastic optimization

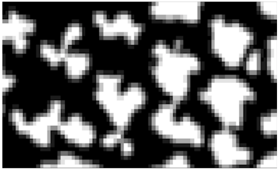


Volume: 0.347

- inequality constraint satisfied: the average of Von Mises function in the solid is under the yield stress

$$\int_\Omega f\left(\sigma^e\right) - \sigma_Y \mathrm{d}\Omega \leq 0$$

Shakedown optimization



Volume: 0.301

- inequality constraint satisfied: the averaged safe-state conditions are satisfied
$$\int_\Omega \left(f(\sigma) - \sigma_Y\right) \mathrm{d}\Omega \leq 0$$
$$\int_\Omega \left(f(\sigma - \sigma^e(\Omega)) - \sigma_Y\right) \mathrm{d}\Omega \leq 0$$
- equality constraint:
$$\int_\Omega (\sigma - \sigma^e(\Omega)) : e(\zeta) \mathrm{d}\Omega$$
$$= -0.198822$$

Preliminary results and future work

Short-term work

- Check the first results + keep on testing to finish debugging the code
- Adjusting the different parameters:
  - initialization
  - descent step
  - coefficient about compliance
- Modifying the algorithm to respect the shakedown constraint for each shape found during the iterative process (by optimizing the other parameters)

Preliminary results and future work

Long-term work

Simplifications during the optimization

- global consideration of the safe-state constraints to be changed:

$$\forall \underline{x} \in \Omega, \quad f(\sigma)(\underline{x}) - \sigma_Y \leq 0 \quad \Rightarrow \quad \int_\Omega \left(f(\sigma)(\underline{x}) - \sigma_Y\right) \mathrm{d}\Omega \leq 0$$

Assumptions while setting the problem

- structure in 2D here $\Rightarrow$ 3D
- 1 cyclic load $\Rightarrow$ N cyclic loads
- cyclic load $\Rightarrow$ constant + cyclic load
- volumetric loads and temperature gradients

References

[1] Grégoire Allaire. *Numerical analysis and optimization: an introduction to mathematical modelling and numerical simulation*. Oxford University Press, 2007.

[2] Grégoire Allaire and François Jouve. Minimum stress optimal design with the level set method. *Engineering analysis with boundary elements*, 32(11):909–918, 2008.

[3] Grégoire Allaire, François Jouve, and Anca-Maria Toader. Structural optimization using sensitivity analysis and a level-set method. *Journal of computational physics*, 194(1):363–393, 2004.

[4] Grégoire Allaire and Marc Schoenauer. *Conception optimale de structures*, volume 58. Springer, 2007.

[5] Dimitri P Bertsekas. *Constrained optimization and Lagrange multiplier methods*. Academic press, 2014.

[6] Allan F Bower. *Applied mechanics of solids*. CRC press, 2009.

[7] J Bree. Plastic deformation of a closed tube due to interaction of pressure stresses and cyclic thermal stresses. *International journal of mechanical sciences*, 31(11):865–892, 1989.

[8] Giovanni Garcea and Leonardo Leonetti. Decomposition methods and strain driven algorithms for limit and shakedown analysis. In *Limit State of Materials and Structures*, pages 19–43. Springer, 2013.

[9] Xu Guo, Wei Sheng Zhang, Michael Yu Wang, and Peng Wei. Stress-related topology optimization via level set approach. *Computer Methods in Applied Mechanics and Engineering*, 200(47):3439–3452, 2011.

References

[10] Anton Mario Bongio Karrman and Grégoire Allaire. Structural optimization using sensitivity analysis and a level-set method, in scilab and matlab. 2009.

[11] Warner Tjardus Koiter. *General theorems for elastic-plastic solids.* North-Holland Amsterdam, 1960.

[12] Jan A König. *Shakedown of elastic-plastic structures*, volume 7. Elsevier, 2012.

[13] Ernst Melan. *Der Spannungszustand eines" Mises-Hencky'schen" Kontinuums bei veränderlicher Belastung.* Hölder-Pichler-Tempsky in Komm., 1938.

[14] Georgios Michailidis. *Manufacturing constraints and multi-phase shape and topology optimization via a level-set method.* PhD thesis, Ecole Polytechnique X, 2014.

[15] Jorge Nocedal and Stephen Wright. *Numerical optimization.* Springer Science & Business Media, 2006.

[16] Stanley Osher and James A Sethian. Fronts propagating with curvature-dependent speed: algorithms based on hamilton-jacobi formulations. *Journal of computational physics*, 79(1):12–49, 1988.

[17] J-W Simon and D Weichert. Interior-point method for lower bound shakedown analysis of von mises-type materials. In *Limit State of Materials and Structures*, pages 103–128. Springer, 2013.

[18] Michael Yu Wang, Xiaoming Wang, and Dongming Guo. A level set method for structural topology optimization. *Computer methods in applied mechanics and engineering*, 192(1):227–246, 2003.

References

[19] K Wiechmann and E Stein. Shape optimization for elasto-plastic deformation under shakedown conditions. *International journal of solids and structures*, 43(22):7145–7165, 2006.

[20] Qi Xia, Tielin Shi, Shiyuan Liu, and Michael Yu Wang. A level set solution to the stress-based structural shape and topology optimization. *Computers & Structures*, 90:55–64, 2012.

[21] N. Vermaak, M. Boissier, L. Valdevit, R. M. McMeeking, "Some Graphical Interpretations of Melan's Theorem for Shakedown Design" In: Direct Methods of Structural Analysis, Editors: Alan Cocks, Olga Barrera, Alan Ponter, Springer 2016 (Submitted).

PROOF OF LOWER BOUND SHAKEDOWN THEOREM

CONTENTS

- Assumptions and preliminary concepts
  - Principle of virtual work
  - Principle of maximum plastic resistance
- Melan’s lower bound shakedown theorem
- Proof of Melan’s theorem

Assumptions:

- All deformations are small and strain field can be derived from deformations as:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

- Body forces (X_i) and surface tractions (p_i) can vary arbitrarily and independently.

- Loads are applied sufficiently slowly so that dynamic effects can be neglected.

Preliminary concepts: Principle of virtual work

- Equilibrium condition is given by the Principle of virtual work:

$$\int \mathrm{X}_i\, u_i dv + \int p_i\, u_i ds = \int \sigma_{ij}\, \varepsilon_{ij} dv$$

which holds for all kinematically admissible strain distributions.

Preliminary concepts: Principle of maximum plastic resistance [1]

- Assume σ_{ij} is a stress state which creates plastic deformation in a structure. Principle of maximum plastic resistance states that, this stress state (σ_{ij}) gives an increment of work that exceeds or equals the work which would be done by a stress state within or at the yield limit (σ_{ij_a}):

$$\left(\sigma_{ij} - \sigma_{ij_a}\right)\dot{\varepsilon}''_{ij} \geq 0$$

σ_{ij_a}: Allowable stress state within or at the yield limit

ε''_{ij} : Plastic strain caused by σ_{ij}

Melan's Lower Bound Shakedown Theorem [2,3]:

- Actual stresses in a plastically deformed body may be written as the sum of fictitious elastic stresses (σ_{ij}^*) and residual stresses (ρ_{ij}) caused by plastic deformation.

$$\sigma_{ij} = \sigma_{ij}^* + \rho_{ij}$$

- If any system of residual stresses can be found such that sum of fictitious elastic stresses and residual stresses constitutes a safe state of stress at every point of the structure and for all possible load combinations, then structure will shakedown and subsequent loads will be carried in a purely elastic manner.

Proof of Melan's Theorem:

- Consider a fictitious strain energy associated with the difference between actual residual stress field (ρ_{ij}) and residual stress field that is assumed to satisfy the theorem ($\bar{\rho}_{ij}$).

$$A = \frac{1}{2}\int (\rho_{ij} - \bar{\rho}_{ij})(\varepsilon'_{ij_r} - \bar{\varepsilon}'_{ij_r})\, dv$$

where ε'_{ij_r} is the elastic strain field corresponding with residual stresses.

- Time derivative of A is (see additional slides for detailed steps):

$$\dot{A} = \int (\rho_{ij} - \bar{\rho}_{ij})\dot{\varepsilon}'_{ij_r}\, dv$$

- Note that actual strains in the structure is a combination of elastic and plastic strains such that:

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij} = \varepsilon^*_{ij} + \varepsilon'_{ij_r} + \varepsilon''_{ij}$$

Elastic strain

Plastic strain

Elastic strain due to fictitious elastic stress

Elastic strain due to residual stress

Plastic strain

Proof of Melan's Theorem (cont'd.):

- Substituting the time derivative of elastic strain due to residual stress we have:

$$\dot{A} = \int (\rho_{ij} - \bar{\rho}_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^*_{ij_r} - \dot{\varepsilon}''_{ij})\, dv$$

which can be rewritten as:

$$\dot{A} = \int (\rho_{ij} - \bar{\rho}_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^*_{ij_r})\, dv - \int (\rho_{ij} - \bar{\rho}_{ij})(\dot{\varepsilon}''_{ij})\, dv$$

- Since $(\rho_{ij} - \bar{\rho}_{ij})$ is a self equilibrating stress state and $(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^*_{ij_r})$ is kinematically admissible since it is derived by subtraction of two kinematically admissible strain fields, Principle of Virtual Work asserts that:

$$\int (\rho_{ij} - \bar{\rho}_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^*_{ij_r})\, dv = 0$$

Proof of Melan's Theorem (cont'd.):

- Time derivative of fictitious strain energy becomes:

$$\dot{A} = -\int (\rho_{ij} - \bar{\rho}_{ij}) \dot{\varepsilon}''_{ij} \, dv$$

- Recall from the theorem that actual stresses may be written as the sum of fictitious elastic stresses (σ^*_{ij}) and residual stresses (ρ_{ij}) caused by plastic deformation which results in:

$$\rho_{ij} = \sigma_{ij} - \sigma^*_{ij}$$

  and the residual stress that is assumed to satisfy the theorem can be written as:

$$\bar{\rho}_{ij} = \sigma_{ij_a} - \sigma^*_{ij}$$

  where σ_{ij_a} is the allowable stress state assumed to exist according to the theorem.

- Substituting these residual stress definitions in the above gives:

$$\dot{A} = -\int (\sigma_{ij} - \sigma_{ij_s}) \dot{\varepsilon}''_{ij} \, dv$$

Proof of Melan's Theorem (cont'd.):

$$\dot{A} = -\int \left(\sigma_{ij} - \sigma_{ij_s}\right)\dot{\varepsilon}''_{ij}\, dv$$

- On the account of principle of maximum plastic resistance $[\left(\sigma_{ij} - \sigma_{ij_a}\right)\dot{\varepsilon}''_{ij} \geq 0]$ this equation states that time derivative of fictitious strain energy ($\dot{A}$) is negative whenever plastic deformation exists.

- However A has to be positive, because strain energy density is always positive or zero [1]. These can only be satisfied simultaneously if either plastic strain rate vanishes or $\sigma_{ij} - \sigma_{ij_s}$ is zero. In either case the structure must shakedown to an elastic state.

[1] Bower, A.F., 2009. *Applied mechanics of solids*. CRC press.
[2] Melan, E., 1938. Zur plastizität des räumlichen kontinuums. *Archive of Applied Mechanics*, *9*(2), pp.116-126.
[3] Koiter, W.T., 1956. A new general theorem on shakedown of elastic-plastic structures. *Proc. Koninkl. Ned. Akad. Wet. B*, *59*, pp.24-34.